Probability and Statistics: A Primer for Beginners and Pre-Beginners

Prologue to the Prologue: Set Theory

Part Three: Set Theory Conclusion

Primary reference: Casella-Berger 2nd Edition

LEVEL UP!

Until now, you've seen sets represented as collections of dice faces, or as events labelled with capital letters. Unfortunately, there are only 26 of those, and we're about to go to infinity and beyond (or at least just to infinity), so from now on, instead of A, B, C, etc., you might be seeing A_1, A_2, A_3, …, etc. Stay vigilant and watch out for those subscripts!

Let's ease into the new notation. Remember our buddies events A and B from Part 1? Well, now they're A_1 and A_2.

$$A_1 = \{x, y\}$$

$$x \in A_1$$

$$y \in A_1$$

$$A_2 = \{x, y, z\}$$

$$x \in A_2$$

$$y \in A_2$$

And so…

$$A_1 \subset A_2$$

Two events are nice, but four really gets the party going! A_1 and A_2, meet your new neighbors, A_3 and A_4!

$$A_1 = \{x, y\}$$

$$A_2 = \{x, y, z\}$$

$$A_3 = \{\mathrm{w}, x, y\}$$

$$A_4 = \{v, w, x, y\}$$

Let's check out the union and intersection of these four events.

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \{v, w, x, y, z\}$$

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \{x, y\}$$

That is a really cumbersome way to write union and intersection operations.
Luckily, we've got a better way.

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=1}^{4} A_i = \{v, w, x, y, z\}$$

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \bigcap_{i=1}^{4} A_i = \{x, y\}$$

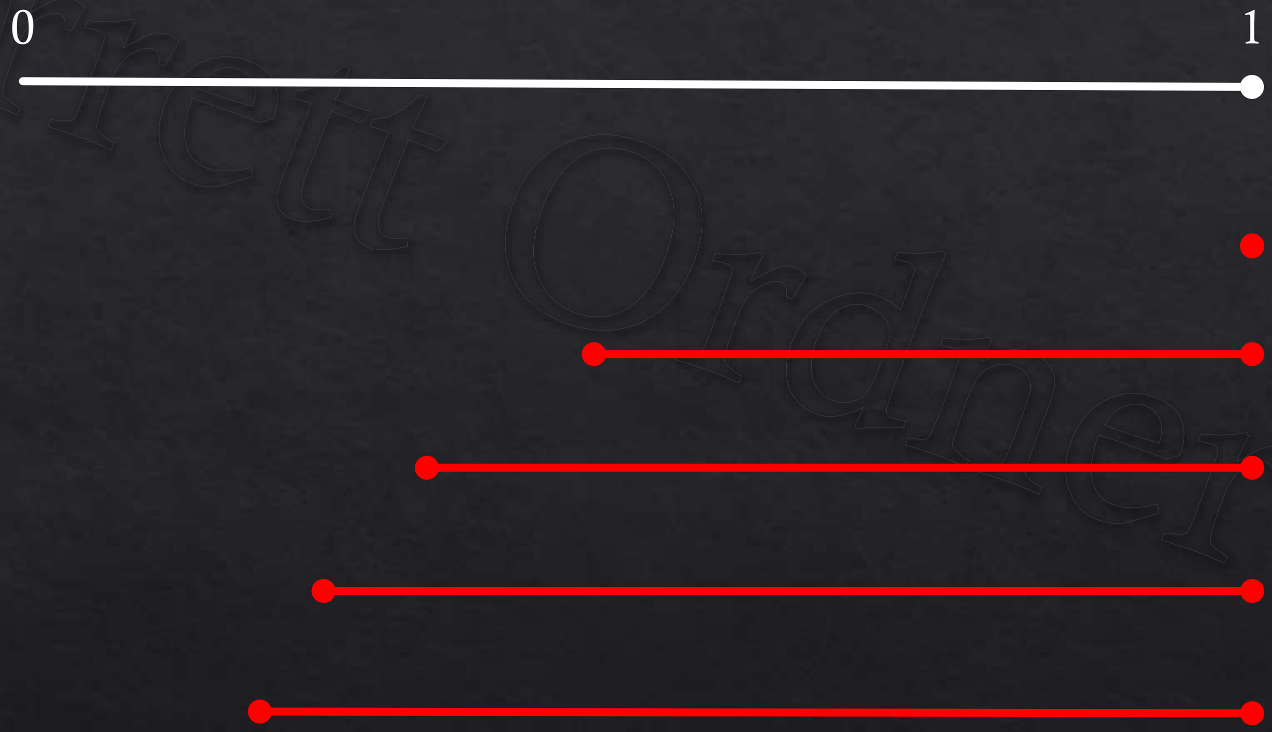
(stopping point)

(index)

(starting point)

Neat, now we can use as many events as we want! We could even use an **infinite** number of events!

Maybe we've got events that aren't so simple. Let's say we've got a sample space Ω = (0,1], and an infinite number of events defined as B_i=[1/*i*,1). For example, B_2=[1/2,1], the closed set of real numbers from 0.5 to 1. Before we get back to the fancy notation, let's draw a (poorly-scaled) picture of what that must look like.



B_1= [1,1]

B_2= [0.5,1]

B_3= [1/3,1]

B_4= [0.25,1]

B_5= [0.2,1]

Notice how every subsequent event is a subset of the event that comes after it? As in, [1, 1] is a subset of [1/2, 1], which is a subset of [1/3, 1], etc.

Now let's write out the union and intersection of all these events:

$$\bigcup_{i=1}^{\infty} B_i = \left[\frac{1}{1}, 1\right] \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \cdots = (0,1]$$

$$\bigcap_{i=1}^{\infty} B_i = \left[\frac{1}{1}, 1\right] \cap \left[\frac{1}{2}, 1\right] \cap \left[\frac{1}{3}, 1\right] \cap \cdots = [1,1] = 1$$

Given the picture we drew on the previous slide, it makes sense that the union of all these events is equal to the event with the most possible outcomes, which approaches (0, 1],* while the intersection of all these events is equal to the event with fewest possible outcomes, [1,1], or 1.

*If you haven't taken a calculus course, read up on "limits" for more info.

Let's take one more look at our picture for an illustration of what the union and intersection look like.

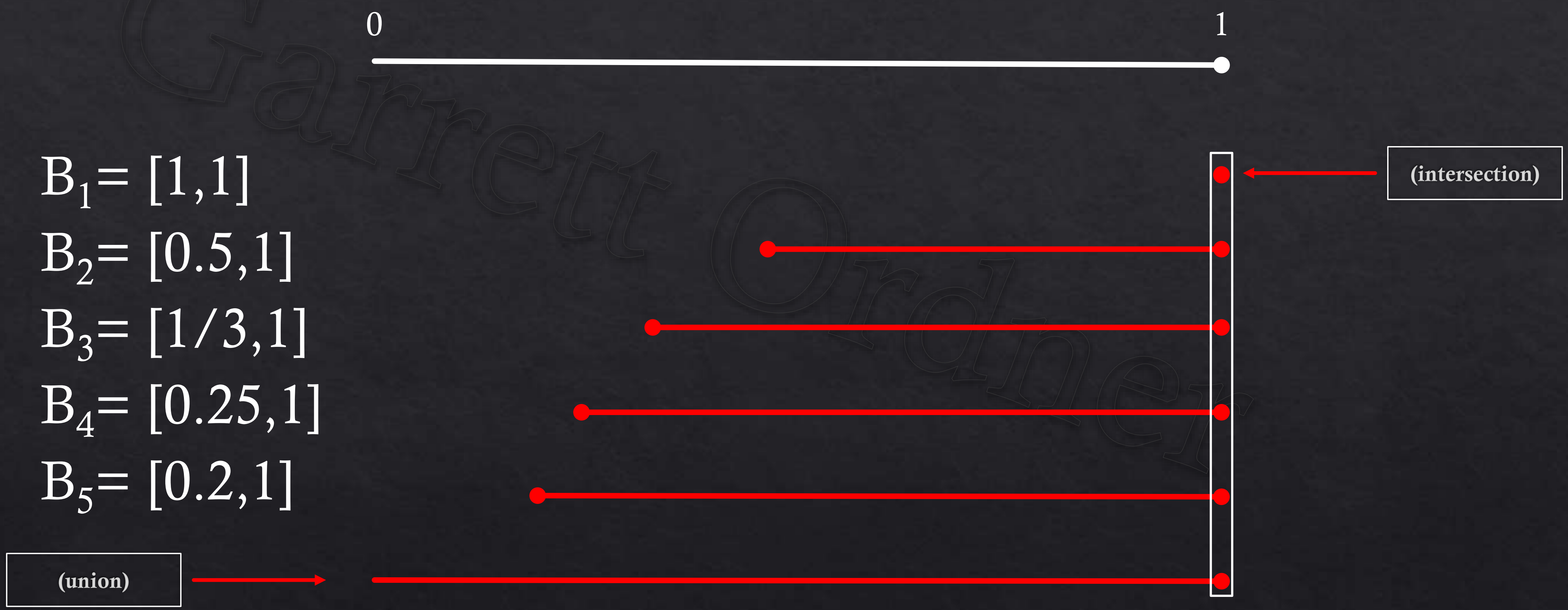
$B_1 = [1,1]$

$B_2 = [0.5,1]$

$B_3 = [1/3,1]$

$B_4 = [0.25,1]$

$B_5 = [0.2,1]$



B_i will continue to approach, though never quite reach, (0,1].

One last quirk of our notation: Remember how the number of possible outcomes in the sample space can be uncountably infinite? Well, so can the number of sets. ☹ Luckily, instead of an index that counts up like *i,* we can just define the union or intersection over some index set (e.g. all real numbers), represented by something like Γ :

$$\bigcup_{a \in \Gamma} C_a$$

Union of sets C_a for every a in Γ.

$$\bigcap_{a \in \Gamma} C_a$$

Intersection of sets C_a for every a in Γ.

We're almost done covering the basics of set theory, but we need to cover a couple of important definitions first. Let's define a small sample space and some events D_i.

$$\Omega = \{a,b,c,d,e,f\}$$

$$D_1 = \{a\}$$

$$D_2 = \{b,c\}$$

$$D_3 = \{d,e\}$$

$$D_4 = \{f\}$$

First, I'm gonna challenge you to write out the union and intersection of these events to prove to yourself that:

$$\bigcup_{i=1}^{4} D_i = \Omega \qquad \bigcap_{i=1}^{4} D_i = \emptyset$$

About the intersection on the right: Not only do the four events we defined not have any element in common, making them **<u>disjoint</u>**, but also for *any pair* of events there are no elements in common, making them **<u>pairwise disjoint.</u>**

Now that we've established the four events are pairwise disjoint, the union on the left gives us the final piece of the puzzle: The union of these pairwise disjoint events is the sample space. In layman's terms, the sample space is divided up amongst these four events.

More formally, the events form a **<u>partition</u>** of the sample space.

COURSE CLEARED!

Woah, you just made it all the way through section 1.1 of Statistical Inference. Now it gets fun. Probability is up next!