Probability and Statistics: A Primer for Beginners and Pre-Beginners

The Journey Begins: Probability Theory

Part One: Of Axioms and Algebras

Primary reference: Casella-Berger 2nd Edition

In the beginning, there was… Ω

(the sample space)

And in it were…

ALL POSSIBLE OUTCOMES!

(of an experiment)

…déjà vu?

But maybe some of those outcomes were more likely than others?

If you rolled a die with these six faces 100 times…

…you might expect a different outcome than you'd get from one with these six faces.

So maybe probability could be interpreted in terms of the frequency of occurrence of certain outcomes of an experiment?

Maybe, but some “experiments” can’t be repeated.

How many times can you repeat an election?

So maybe interpretation of probability is more subjective?

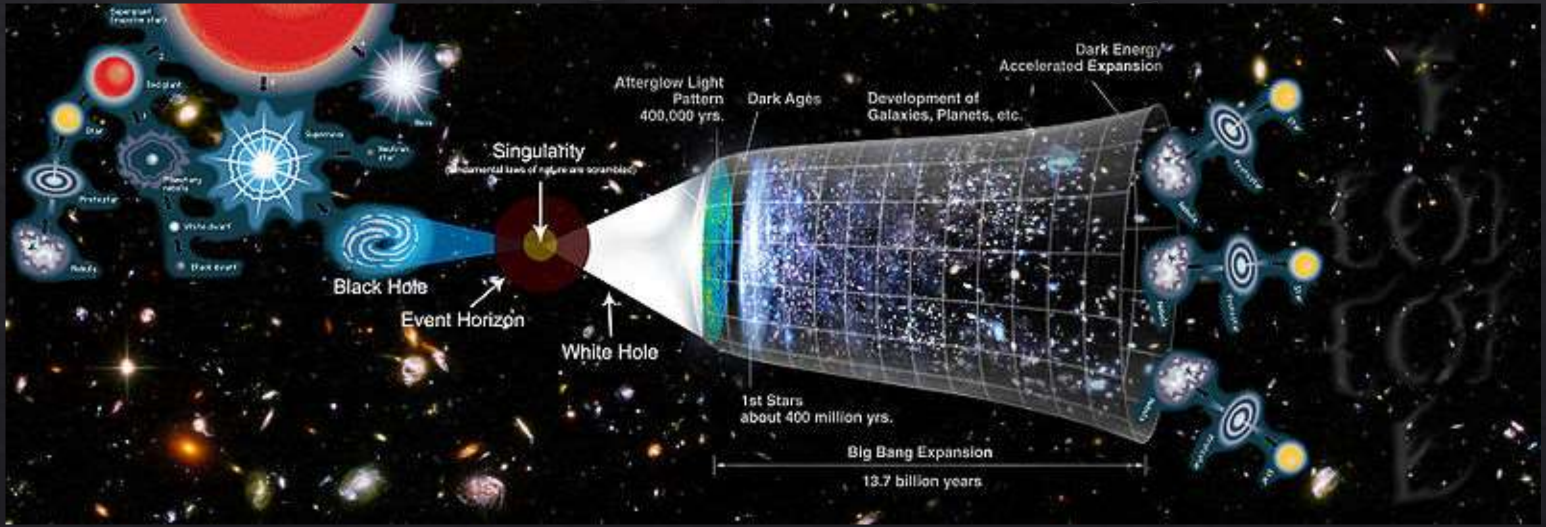
Doesn't matter, it's math, baby!

We don't need no stinking context
when we got cold, hard axioms!

For now, we don't need to interpret
probabilities, we just gotta make
sure they follow the rules.

Doesn’t matter, it’s math, baby!

But before we can talk about the axioms of probability, first we need to talk about parallel universes.



(nah just sigma algebras)

The humble σ-algebra (a.k.a. Borel field)

First, it gets a fancy letter: $\mathcal{B}$

Second, it's a term for a collection of subsets of our sample space Ω that has special properties!

The special rules of our humble Borel field

a. $\emptyset \in \mathcal{B}$: It ain't a Borel field unless the empty set is an element!

b. For any set A, if $A \in \mathcal{B}$, then $A^C \in \mathcal{B}$. Remember, A^C is the *complement* of A! This property is known as being *closed under complementation*!

c. If sets $A_1, A_2, A_3, \ldots \in \mathcal{B}$, then
$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$$
This property is known as being *closed under countable unions*!

Simple enough, but we can stretch them a bit!

a. $\emptyset \in \mathcal{B}$

So the empty set is in $\mathcal{B}$. Big deal, right? $\emptyset$ is a subset of *every* set.

But, $\emptyset = \Omega^c$, and rule b. said $\mathcal{B}$ had to be *closed under complementation*! So now Ω is always in $\mathcal{B}$, too!

The special rules of our humble Borel field

c. If sets $A_1, A_2, A_3, \ldots \in \mathcal{B}$, then
$\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

Oh boy, things get tricky for a bit. By rule b. (closed under complementation), $\left(\bigcup_{i=1}^{\infty} A_i\right)^C \in \mathcal{B}$ too! Plus, $A_1^C, A_2^C, A_3^C, \ldots \in \mathcal{B}$. Applying rule c. once more means $\bigcup_{i=1}^{\infty} A_i^C \in \mathcal{B}$, and a final application of rule b. means $\left(\bigcup_{i=1}^{\infty} A_i^C\right)^C \in \mathcal{B}$!

(is the room spinning, or just my head?!)

It gets worse before it gets better…

Remember DeMorgan's Law? It states that for any sets A_1 and A_2,
$(A_1 \cup A_2)^C = A_1^C \cap A_2^C$.

Well, that law can be shown to prove that
$\left(\bigcup_{i=1}^{\infty} A_i^C\right)^C = \bigcap_{i=1}^{\infty} A_i$, so now the intersection of all those sets is by definition an element of $\mathcal{B}$.

Taking stock of everything in the Borel field:

1. $\emptyset \in \mathcal{B}$
2. $\Omega \in \mathcal{B}$
3. If sets $A_1, A_2, A_3, \ldots \in \mathcal{B}$,
   a) $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$
   b) $\left(\bigcup_{i=1}^{\infty} A_i\right)^C \in \mathcal{B}$
   c) $A_1^C, A_2^C, A_3^C, \ldots \in \mathcal{B}$
   d) $\bigcup_{i=1}^{\infty} A_i^C \in \mathcal{B}$
   e) $\left(\bigcup_{i=1}^{\infty} A_i^C\right)^C = \bigcap_{i=1}^{\infty} A_i \in \mathcal{B}$

What does this all mean? Maybe not much…

For all of that misery, we're only interested in the smallest Borel field that contains all the open sets of Ω. If the elements of Ω are finite or at least countable, then this is just all the subsets of Ω, including Ω itself.

What does this all mean? Maybe not much…

Ok, so let’s define a little sample space again, and figure out which sets go into our Borel field.

$$\Omega = \{x, y, z\}$$

1. $\emptyset$
2. $\{x, y, z\}$
3. $\{x\}$
4. $\{y\}$
5. $\{z\}$
6. $\{x, y\}$
7. $\{x, z\}$
8. $\{y, z\}$

What if Ω is uncountable?

…you don't want to know.

I didn't come here to learn about no stinking Borel fields! I want to learn the axioms of probability!

And so you shall, my friend! For now, I shall unveil to you Kolmogorov's axioms of probability, though only to whet your appetite for the decadent mathematical pleasures yet to come!

Now for the moment you've been waiting for, here they are!

Axioms of Probability

1. $\mathrm{P}(A) \geq 0$ for all $A \in \mathcal{B}$
2. $\mathrm{P}(\Omega) = 1$
3. If sets $A_1, A_2, A_3, \ldots \in \mathcal{B}$ are pairwise disjoint, then $\mathrm{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

(okay bye)