Probability and Statistics: A Primer for Beginners and Pre-Beginners

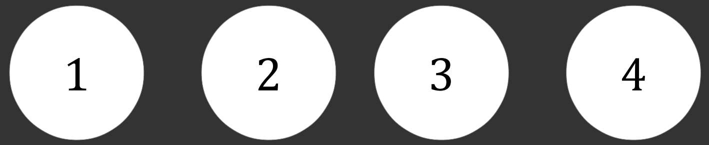
The Journey Continues: Conditional Probability and Independence

Part One: Conditional Probability

Primary reference: Casella-Berger 2nd Edition

Unconditional Probability

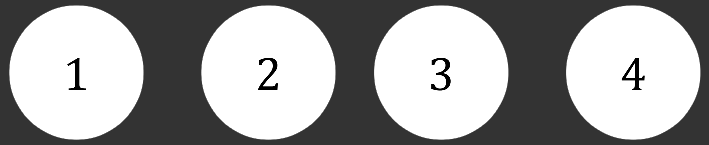
Until now, we've been looking only at unconditional probabilities. Let's use our lottery balls as an example. This time, lets look at picking two balls out of four:



We know that the probability of picking a given number is ¼, or 0.25. But what is the probability of picking two specific numbers? For example, what is the probability of picking the two odd numbers?

Unconditional Probability

We can find this probability using the technique from lesson 1-2-7: Divide the number of outcomes in which we pick the odd number with the number of total possible outcomes.

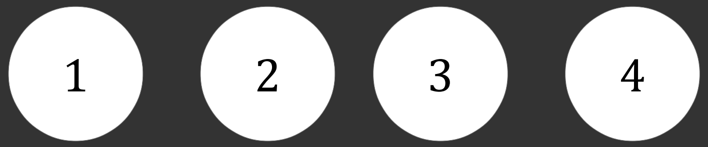


So, if we are picking two balls, lets define the possible outcomes in our sample space.

Defining the sample space Ω

Let our sample space be all combinations of two balls picked from the four possibilities (order is not important):

$$\Omega = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$



Notice we have six possible combinations, and only (1,3) fits our criteria, so the probability of picking the two odd numbers is 1/6.

Another perspective

So, we've seen how we can calculate the probability of picking the odd numbers using counting, but there's another way: What if we did this in two steps:

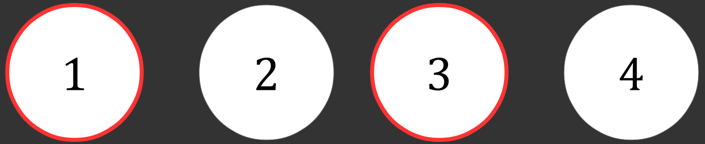
1. Find the probability of picking an odd number on the first pick.

2. Find the probability of picking an odd number on the second pick, *conditional* on our having picked an odd number in step one.

Lets use a visual example to illustrate this.

Another perspective

Our sample space for the first pick is the four balls, and two of the balls have an odd number, so the probability of picking an odd number is 2/4 = 1/2:



Let's say we got lucky and picked 1. Now we want to find the probability of picking 3. However, our sample space looks different now:

Another perspective

Now we only have one odd number to pick out of three possible choices. So, the probability of picking an odd number *given that the first pick was odd* is 1/3.

We'd say the conditional probability of an odd second pick given an odd first pick is 1/3.

Time for some notation

We can write out this conditional probability. Let the events A = {odd number on first pick} and B = {odd number on second pick}. We would express the conditional probability of an odd second pick *given* an odd first pick as P(B|A).

We can actually calculate conditional probabilities as a ratio of unconditional probabilities:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Remember that P(B∩A) is the probability of picking an odd number on both picks.

Calculating our conditional probability

Remember from our earlier calculations that P(A) = 2/4, and P(B∩A) = 1/6.

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{2}{4}} = \frac{4}{12} = \frac{1}{3}$$

Still, the conditional probability of picking an odd number on the second try given an odd first pick is slightly easier in this case to figure out than the unconditional probability of two odd picks. Maybe we can rearrange this formula to something more helpful.

Calculating the unconditional probability

If we rearrange the formula, we see we can use P(A) and P(B|A), which were both easy to find, to calculate the unconditional probability of picking two odd numbers.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \rightarrow P(B \cap A) = P(B|A)\,P(A)$$

$$= (1/3) * (2/4) = \frac{1}{6}$$